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PERSONAL REMINISCENCES AND REMARKS ON THE MATHEMATICAL WORK OF TIBOR GALLAI

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Tibor Gallai is my oldest friend. We met more than 52 years ago, and we had our first long mathematical discussion in the summer of 1930. Our friendship and collaboration started then, and no doubt will last until one of us "leaves". I still remember that I learned from him at this occasion the beautiful proof of Jacob Steiner for the isoperimetric property of the circle. He pointed out to me that the proof is not quite accurate since Steiner only proves that if the maximum exists then it must be the circle. About two years later I proved that Steiner's proof can easily be repaired since one can show that it can be made to converge. Soon after, I found that Caratheodory found the same proof in 1909 and published it in the Mathematische Annalen.

I knew of Gallai's existence many years before we met, since we both were diligent and successful solvers of the problem section of the Hungarian Mathematical Journal for High Schools. We both and in fact most of the Hungarian mathematicians (Gallai on the whole was better at it than I, but Hajós and later Vázsonyi were the real stars) of our generation owe a great deal to this journal and its illustrious editor Andor Faragó, who by the way studied mathematics with my parents and who was murdered together with his children by the Fascists in 1945.

Gallai's parents were not very well off. Thus he often had to tutor students and rarely could devote himself completely to mathematics. As a Jew, he faced immediately a serious hurdle; by a law passed in 1920 only 5% of the university students could be Jewish. Thus it was essential that he should win the mathematical competition for high school students held in June since the victor was automatically admitted to university. I am fairly sure Gallai would have easily won the competition but due to the ill will of some of his teachers he was not allowed to take part. He at first tried to get in to the technological university to become an engineer. I very much disapproved of this since I wanted him to study mathematics. "Fortunately" he was not admitted and after he won the Eötvös competition* held in the early autumn of 1930, he was admitted to the University and started his brilliant career as a mathematician.

* Mathematics competition for students after matriculation in Hungary. Cf. *Hungarian Problem Book I-II* (ed. G. Hajós, G. Neukomm and J. Surányi), New York, Random House, 1963.

We shared a common interest in graph theory, set theory, combinatorics and geometry.

We both went to the lectures of Dénes Kőnig on graph theory*. In those "dark and uncivilised" times, graph theory and combinatorics were held in amused contempt by most mathematicians. A friend of my parents once said: "Dénes Kőnig is great in his art but his art is so small", and the great English topologist J. H. C. Whitehead referred to graph theory as the "slums of topology". Now I think most mathematicians would remember J. Kőnig — an eminent set theorist — first of all as the father of Dénes Kőnig, which was very different in 1930.

Gallai's great ability became evident immediately. As a freshman he proved that the graph whose vertices are the lattice points in 3-dimensional space, two lattice points are joined if they differ in only one coordinate by one, is both Hamiltonian and Eulerian. This led us later to our first joint paper. Gallai, Vázsonyi and I gave a necessary and sufficient condition for an infinite graph to have an Euler line. Due mainly to the work of Nash-Williams, this paper is now only of historical interest.

Very soon afterwards he proved the pretty equality**

$$P_{\min}+P_{\max} = E_{\min}+E_{\max} = n$$

which is valid for every graph of n vetrices. At that time, we discussed various generalizations and extensions. We never published these results, they have been later covered by the results of L. Lovász.

He suggested to me more than 50 years ago that structures which are now known as hypergraphs should be studied—nothing came of this until much later.

Also as freshman he proved that if $\{I_k\}$ is a family of intervals on the real axis then the maximum number of disjoint intervals of the family equals the minimum number of points $y_1, \dots, y_l$ so that every interval of the family $\{I_k\}$ contains at least one of the points y_i.

Unfortunately he was often very slow in publishing, and several of his results were later independently discovered and published by others. I promised Gallai long ago never to state these facts in his lifetime. Thus this discussion has to be finished at a less happy occasion I hope in the distant future.

In 1933 while reading the beautiful book "Anschauliche Geometrie" of Hilbert and Cohn-Vossen (the English translation is entitled "Geometry and the imagination") the following pretty conjecture occurred to me: Let $x_1, \dots, x_n$ be a fiinite set of points in the plane not all on a line. Then there always is a line which goes through exactly two of the points. I expected this to be easy but to my great surprise and disappointment I could not find a proof. I told this problem to Gallai who very soon found an ingenious proof. L. M. Kelly noticed about 10 years later that the conjecture was not new. It was first stated by Sylvester in the Educational Times in 1893. The first proof though is due to Gallai. For a detailed history of the problem see the papers of Motzkin and Grünbaum ([13], [10]). The result of Gallai was the starting point of many papers by Edelstein and others. A very recent paper of Borwein [1] shows that the subject is still very much alive.

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* Kőnig was a lecturer of the Technical University where engineers are trained.

** $P_{\min}$ and $E_{\min}$ are the minimum number of vertices (edges) covering every edge (vertex, resp.). $P_{\max}$ and $E_{\max}$ are the maximum number of independent vertices (edges, resp.).

In this short note I will mainly discuss results of Gallai which are related to my own and which are not so well known. Thus first of all I discuss his work on polynomials. We begin with his dissertation [11] which unfortunately only appeared in Hungarian.

The starting point was the following theorem of Schur: Let $f(x)$ be a polynomial of degree n, all of whose roots are real. Denote by x_k, $0 \leqq k \leqq n-1$ the largest root of $f^{(k)}(x)$ (the k-th derivative). Then

$$x_0 - x_1 \leqq x_1 - x_2 \leqq \ldots \leqq x_{n-2} - x_{n-1}.$$

Gallai proved the following sharper result: Put

$$(1) \qquad g(x) = A(x-a_1)\ldots(x-a_n), \quad n \geqq 2, \quad A < 0, \quad a_1 > a_2 \geqq a_3 \geqq \ldots \geqq a_n.$$

Then if b_1 denotes the largest root of $g'(x)$ we have

$$\int_{a_2}^{a_1+(a_1-b_1)} g(x)\,dx \leqq 0.$$

The second main theorem of the thesis was influenced by an old result of Laguerre and Gyula Sz.-Nagy. Gallai proved the following theorem:

Let $f(x)$ be a polynomial of degree n, p and q $(p<q)$ two consecutive real roots. Assume further that all roots of $f'(x)$ are real. Then neither of the intervals

$$\left(p, p+\frac{q-p}{n}\right) \quad \text{and} \quad \left(q-\frac{q-p}{n}, q\right)$$

can contain all the real roots of $f'(x)$ which fall in the interval (p, q).

Gallai asks the following question: Assume that all roots of $f'(x)$ are real and that $f'(x)$ has at most one root in (p, q). Is it then true that $f(x)$ has no root in the circle of diameter* (p, q)?

Finally let $g(x)$ be of the form (1) and b_1 the largest root of $g'(x)$. I proved that

$$(2) \qquad \int_{a_2}^{b_1} |g(x)|\,dx < 4 \int_{b_1}^{a_1} |g(x)|\,dx.$$

Gallai proved that in (2) the constant 4 can be replaced by $2\left(\left(\frac{n}{n-1}\right)^n - 2\right)^{-1}$ and that this is best possible. In fact he proves an analogous best possible result for all intervals (a_i, a_{i-1}) and several further related results.

I hope the reader will forgive me the detailed discussion of these results at the expense of more important later results of Gallai. I had two reasons: First it shows that Gallai did nice work outside combinatorial analysis, and this paper appeared only in Hungarian (with German summary) and was completely forgotten.

Under the influence of these results Gallai and I proved the following inequal-

* That is, the circle of radius $(q-p)/2$ with centre $(p+q)/2$.

ity: Let $f(x)$ be a polynomial with only real roots. Let $f(-1)=f(+1)=0$ be consecutive roots and $\max_{-1\le x\le 1} f(x)=1$. Then

$$\int_{-1}^{1} f(x)\,dx \le 4/3. \tag{3}$$

We proved several related results. On p.1 of Vol.1 of the Math. Reviews Pólya writes a very nice review, he calls (3) a generalization of a theorem of Archimedes and simplifies some of the proofs by using a method of Szekeres.

As far as I know Gallai never returned to the study of polynomials after this paper.

Here is another nice problem of Gallai which attracted quite a lot of attention: Let $C_1, \dots, C_n$ be a family of circles every two of which has a non-empty intersection. Gallai conjectured that then there are always 4 points in the plane so that every circle contains at least one of them. This conjecture was finally proved by Stacho and Danzer (Danzer's proof was never published). Gallai then posed the following problem: Let $f(k)$ be the smallest integer so that if $\{C_i\}$ is a system of circles such that there are at most k circles having pairwise empty intersection, then there are at most $f(k)$ points $x_1, \dots, x_{f(k)}$ so that each of our circles contains at least one of these points. Determine (or estimate) $f(k)$. Although $f(1)=4$, $f(2)$ is unknown.

When we were freshmen, Kőnig in his lecture on graph theory suggested that Menger type theorems may hold in several branches of mathematics. Gallai informs me that the above problem and several others which I already mentioned were motivated by this suggestion of Kőnig.

Here is another example. Gallai asked the following question: Let $h(n)$ be the smallest integer for which every graph in which the maximal number of vertex disjoint circuits is n, contains $h(n)$ vertices so that every circuit goes through at least one of these $h(n)$ vertices. Determine $h(n)$. Bollobás proved $h(1)=3$, and Pósa and I proved ([9])

$$c_1 n \log n < h(n) < c_2 n \log n.$$

Gallai also asked the same question for directed graphs, but as far as I know this question has not yet been investigated.

Now let me return to a short review of our joint work with Gallai. We wrote four joint papers since 1959. The one which is quoted most is perhaps the least deep and is really essentially contained in a result of Tutte: We obtained the necessary and sufficient conditions that to a set of n integers $v_1 \ge v_2 \ge \dots \ge v_n$ there should exist a graph of n vertices $x_1, \dots, x_n$ where the degree (or valency) of x_i should be v_i. Several mathematicians including ourselves tried to extend our theorem to hypergraphs, but no satisfactory solution has been found so far.

In our other joint paper we investigate the following problem: Denote by $f(n;k)$ and $F(n;k)$ the smallest integer for which every graph of n vertices and $f(n;k)$ (resp. $F(n;k)$) edges contains a path (resp. a circuit) of length k. We determine $f(n;k)$ and $F(n;k)$ nearly exactly.

In our second long paper we investigate various relations between $v(G)$, $e(G)$ and $\tau(G)$ where $v(G)$ is the number of vertices, $e(G)$ the number of edges and

$\tau(G)$ is the smallest set of vertices which represent every edge of G. We prove among others that

$$\tau(G) \leqq \frac{2}{\frac{1}{\frac{1}{2}v(G)}+\frac{1}{e(G)}}.$$

Further if $\tau(G')\leqq p$ for every subgraph G' of G satisfying $v(G')\leqq 2p+2$, then $\tau(G)\leqq p$.

We also obtain various inequalities for hypergraphs and state some conjectures which were settled in a paper of Hajnal, Moon and myself [8].

A little known (unpublished) result of Gallai states that the vertices of every graph can be decomposed into two classes so that the graphs induced by the two classes have all their vertices of even degree. Gallai's original proof used linear algebra. Later Pósa found a very ingenious direct combinatorial proof (see problem 5.17. in [12]).

In our last joint paper, we settle a question of Dirac posed at the graph theory conference in Smolenice in 1963. We prove that if $G(n)$ is a regular graph of n vertices of valency $h<n-1$ then its chromatic number is less than or equal to $\frac{3n}{5}$.

I do not want to write a great deal more about Gallai's work in combinatorics since this is discussed in the paper of Lovász. I just remark that perhaps his most important work deals with combinatorial max-min theorems, critical graphs of various sorts and factorizations.

Gallai had two students, B. Andrásfai and G. Hetyei*, whom he prepared for the degree of candidacy. This degree requires slightly more research than the Ph. D. at most universities. Pósa and Lovász can be also considered to be his students, and of course he had many more of them through his papers.

Now I want to talk about Gallai as a human being and a teacher. He is very modest, I would say abnormally so. His friends (myself included) failed so far to persuade him to accept the degree of "Doctor of the Academy" (this is at least as valuable as a D. Sc. in Britain). He claims that he does not deserve it. He surely is in a minority of one in this opinion. Here is another true story. In March 1956 he got the Kossuth prize (at that time it was 20 000 forints, about a year's salary of a teacher). There was a bad flood in Hungary at about that time, and money was collected for the victims. Gallai gave 25 000 forints, nobody else gave more than 1000. He could not afford this generosity very well, but, as always, he obeyed what his conscience dictated.

He is not only a first rate mathematician but also an excellent teacher. From 1946 to 1950 he taught in the Jewish high school for girls in Budapest. In one year he had 22 students. Six of them became mathematicians and one of them, Vera T. Sós, became one of the leading mathematicians in Hungary. From 1950 to 56 he was Professor at the Technical University in Budapest and was one of the most popular

* For references of some of their work, cf. Lovász' book [12].

and successful teachers. In spite of the fact that he was fairly severe at the examination he was greatly beloved by his students.

He and Rózsa Péter wrote a very interesting and excellent textbook of mathematics for high school students.

He was and is always ready to help his colleagues and students whenever they need advice or help in mathematical or personal matters.

We all hope that he will long continue to be active mathematically, and I certainly hope that our last joint paper has not yet been written.

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